Unbiased estimation of the reciprocal of population size

in a k-sample mark-recapture experiment:

Preliminary report

D. S. Robson

BU-378-M June, 1971

ABSTRACT

An unbiased estimator of the reciprocal of population size exists for the two-sample mark-recapture experiment, and we offer the conjecture here that a unique MVUE of $1/N$ also exists for the k-sample experiment in which all unmarked elements in a sample are marked before returning the (entire) sample to the population. A minimal sufficient statistic in this case where the population is assumed to be closed to mortality and recruitment is the number U of district elements in the k samples,

$$U = U_1+U_2+ \cdot\cdot +U_k$$

where U_i is the number of unmarked elements captured in the i'th sample. This conjecture is supported here by a numerical example showing the construction of such an estimator in the case $k = 3$. The estimator is shown to be virtually the same as the reciprocal of the maximum likelihood estimator of N in this case.

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Introduction

An unbiased estimator of the reciprocal of population size exists for the two-sample mark-recapture experiment, and we offer the conjecture here that a unique MVUE of $1/N$ also exists for the k-sample experiment in which all unmarked elements in a sample are marked before returning the (entire) sample to the population. A minimal sufficient statistic in this case where the population is assumed to be closed to mortality and recruitment is the number U of district elements in the k samples,

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where U_i is the number of unmarked elements captured in the i'th sample. This conjecture is supported here by a numerical example showing the construction of such an estimator in the case $k = 3$. The estimator is shown to be virtually the same as the reciprocal of the maximum likelihood estimator of N in this case.

Construction of the MVUE of $1/N$ when $n_0 = 3$, $n_1 = 2$, $n_2 = 3$

$$P_N(U_1 = u_1) = \binom{n_0}{n_1-u_1}\binom{N-n_0}{u_1}\Big/\binom{N}{n_1} \qquad P_N(U_2 = u_2|U_1 = u_1) = \binom{n_0+u_1}{n_2-u_2}\binom{N-n_0-u_1}{u_2}\Big/\binom{N}{n_2}$$

$$= \binom{3}{2-u_1}\binom{N-3}{u_1}\Big/\binom{N}{2} \qquad = \binom{3+u_1}{3-u_2}\binom{N-3-u_1}{u_2}\Big/\binom{N}{3}$$

N = 3

$$P_3(U_1 = 0) = \binom{3}{2}\binom{0}{0}\Big/\binom{3}{2} = 1 \quad P_3(U_2 = 0|U_1 = 0) = \binom{3}{3}\binom{0}{0}\Big/\binom{3}{3} = 1$$

N = 4

$P_N(u_2|u_1)$

u_1	$P_N(u_1)$	$u_2 =$ 0	1
0	1/2	1/4	3/4
1	1/2	1	0

N = 5

		0	1	2
0	3/10	1/10	6/10	3/10
1	6/10	4/10	6/10	0
2	1/10	1	0	0

N = 6

		0	1	2	3
0	1/5	1/20	9/20	9/20	1/20
1	3/5	4/20	12/20	4/20	0
2	1/5	10/20	10/20	0	0

$P_N(u_2|u_1)$

$\underline{N = 7}$

u_1	$P_N(u_1)$	$u_2 = 0$	1	2	3
0	1/7	1/35	12/35	18/35	4/35
1	4/7	4/35	18/35	12/35	1/35
2	2/7	10/35	20/35	5/35	0

$\underline{N = 8}$

		0	1	2	3
0	3/28	1/56	15/56	30/56	10/56
1	15/28	4/56	24/56	24/56	4/56
2	10/28	10/56	30/56	15/56	1/56

Dist'n of $U = n_0+U_1+U_2 = 3+U_1+U_2$

$\underline{N = 3}$ $\quad P_3(U = 3) = 1$

$\underline{N = 4}$ $\quad P_4(U = 3) = P_4(U_1 = 0,\ U_2 = 0) = 1/2(1/4) = 1/8$

$$P_4(U = 4) = P_4(U_1 = 0,\ U_2 = 1) + P_4(U_1 = 1,\ U_2 = 0)$$

$$= 1/2(3/4) + 1/2(1) = 7/8$$

$\underline{N = 5}$

u =	3	4	5
$P_5(u)$ =	3/100	42/100	55/100

$\underline{N = 6}$

u	3	4	5	6
$P_6(u)$	1/100	21/100	55/100	23/100

$\underline{N = 7}$

u	3	4	5	6	7
$P_7(u)$	1/245	28/245	110/245	92/245	14/245

$\underline{N = 8}$

u	3	4	5	6	7	8
$P_8(u)$	3/1568	105/1568	550/1568	690/1568	210/1568	10/1568

<u>Construction of unbiased estimator of 1/N</u>

$$E\,\hat{N}^{-1}(U) = N^{-1}$$

$\underline{N = 3}$

$$\sum_u \hat{N}^{-1}(u)\,P_3(u) = \hat{N}^{-1}(3) = \boxed{\frac{1}{3}}$$

$\underline{N = 4}$

$$\sum_u \hat{N}^{-1}(u)\,P_4(u) = \hat{N}^{-1}(3)\,P_4(3) + \hat{N}^{-1}(4)\,P_4(4) = \frac{1}{4}$$

$$= \frac{1}{3}\left(\frac{1}{8}\right) + \hat{N}^{-1}(4)\cdot\frac{7}{8} = \frac{1}{4}$$

$$\hat{N}^{-1}(4) = \frac{8}{7}\left[\frac{1}{4} - \frac{1}{24}\right] = \boxed{\frac{5}{21}}$$

$\underline{N = 5}$

$$\frac{1}{3}\,P_5(3) + \frac{5}{21}\,P_5(4) + \hat{N}^{-1}(5)\,P_5(5) = \frac{1}{5}$$

$$\frac{1}{3}\left(\frac{3}{100}\right) + \frac{5}{21}\left(\frac{42}{100}\right) + \hat{N}^{-1}(5)\cdot\frac{55}{100} = \frac{1}{5}$$

$$\hat{N}^{-1}(5) = \frac{100}{55}\left[\frac{1}{5} - \frac{1}{100} - \frac{10}{100}\right] = \boxed{\frac{9}{55}}$$

$\underline{N = 6}$

$$\frac{1}{3}\left(\frac{1}{100}\right) + \frac{5}{21}\left(\frac{21}{100}\right) + \frac{9}{55}\left(\frac{55}{100}\right) + \hat{N}^{-1}(6)\cdot\frac{23}{100} = \frac{1}{6}$$

$$\hat{N}^{-1}(6) = \frac{100}{23}\left[\frac{1}{6} - \frac{1}{300} - \frac{5}{100} - \frac{9}{100}\right] = \boxed{\frac{7}{69}}$$

<u>N = 7</u> $\hat{N}^{-1}(7) = \frac{245}{14}\left[\frac{1}{7} - \frac{1}{245}\left(\frac{1}{3} + \frac{20}{3} + 18 + \frac{28}{3}\right)\right] = \boxed{\frac{1}{21}}$

<u>N = 8</u> $\hat{N}^{-1}(8) = \frac{1568}{10}\left[\frac{1}{8} - \frac{1}{1568}\left(1 + 25 + 90 + 70 + 10\right)\right] = \boxed{0}$

u	$\hat{N}^{-1}(u)$
3	1/3 = .3333
4	5/21 = .2381
5	9/55 = .1636
6	7/69 = .1014
7	1/21 = .0476
8	0

<u>Maximum likelihood estimator of 1/N</u>

Likelihood eq'n: $u = N\left[1 - \prod_{0}^{k}\left(1 - \frac{n_i}{N}\right)\right]$ <u>k = 2</u>

$$N_{ML}^{-1} = \frac{\sum_{i<j} n_i n_j - \sqrt{\left(\sum_{i<j} n_i n_j\right)^2 - 4\left(\sum_{0}^{2} n_i - u\right)\prod_{0}^{2} n_i}}{2\prod_{0}^{2} n_i}$$

$$= \frac{21 - \sqrt{441 - 72(8 - u)}}{36}$$

u	N_{ML}^{-1}
3	1/3 = .3333
4	.2397
5	1/6 = .1667
6	.1046
7	.0496
8	0

<u>N = 10</u>

u_1	$P_N(u_1)$	u_2 =	0	1	2	3
0	1/15		1/120	21/120	63/120	35/120
1	7/15		4/120	36/120	60/120	20/120
2	7/15		10/120	50/120	50/120	10/120

u	$15(120)P_N(u)$	$\hat{N}^{-1}(u)$	$15(120)P_N(u)\hat{N}^{-1}(u)$
3	1	1/3	1/3
4	49	5/21	35/3
5	385	9/55	189/3
6	805	7/69	245/3
7	490	1/21	70/3
8	70	0	0

$$\text{Total} = \frac{540}{3} = 180$$

$$E_{10}\hat{N}^{-1}(U) = \frac{180}{15(120)} = \frac{180}{1800} = \frac{1}{10}$$